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## A note on the orthogonal basis of a certain full symmetry class of tensors

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### Abstract

It is shown that the full symmetry class of tensors associated with the irreducible character  $[2, 1^{n-2}]$  of  $S_n$  does not have an orthogonal basis consisting of decomposable symmetrized tensors.

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### 1. Introduction and preliminaries

Let  $V$  be an  $m$ -unitary space. Let  $\otimes^n V$  be the  $n$ th tensor power of  $V$  and write  $v_1 \otimes \cdots \otimes v_n$  for the decomposable tensor product of the indicated vectors. To each permutation  $\sigma$  in  $S_n$  there corresponds a unique linear operator  $P(\sigma): \otimes^n V \rightarrow \otimes^n V$  determined by  $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ . Let  $G$  be a subgroup

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of  $S_n$  and let  $\text{Irr}(G)$  be the set of all the irreducible complex characters of  $G$ . It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi): \overset{n}{\otimes} V \rightarrow \overset{n}{\otimes} V \mid T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma), \chi \in \text{Irr}(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of  $\overset{n}{\otimes} V$  under  $T(G, \chi)$  is called the *symmetry class of tensors* associated with  $G$  and  $\chi$  and it is denoted by  $V_\chi^n(G)$ . The image of  $v_1 \otimes \cdots \otimes v_n$  under  $T(G, \chi)$  is denoted by  $v_1 * \cdots * v_n$  and it is called a *decomposable symmetrized tensor*.

The inner product on  $V$  induces an inner product on  $\overset{n}{\otimes} V$  whose restriction to  $V_\chi^n(G)$  satisfies

$$\langle u_1 * \cdots * u_n \mid v_1 * \cdots * v_n \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n \langle u_i \mid v_{\sigma(i)} \rangle. \tag{1}$$

Let  $\Gamma_m^n$  be the set of all sequences  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $1 \leq \alpha_i \leq m$ . Then the group  $G$  acts on  $\Gamma_m^n$  by  $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)})$ , where  $\sigma \in G$  and  $\alpha \in \Gamma_m^n$ . Let  $O(\alpha) = \{\sigma \cdot \alpha \mid \sigma \in G\}$  be the *orbit* of  $\alpha$ , and  $G_\alpha$  be its *stabilizer subgroup*, i.e.,  $G_\alpha = \{\sigma \in G \mid \sigma \cdot \alpha = \alpha\}$ , and consider a system  $\Delta$  of distinct representatives of the  $G$ -orbits on  $\Gamma_m^n$ .

Suppose  $\{e_1, \dots, e_m\}$  is an orthonormal basis of  $V$ . For  $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_m^n$ , denote by  $e_\alpha^*$  the decomposable symmetrized tensor  $e_{\alpha_1} * \cdots * e_{\alpha_n}$ . Then, by (1), one can easily obtain that for each  $\alpha, \beta \in \Gamma_m^n$ ,

$$\langle e_\alpha^* \mid e_\beta^* \rangle = \begin{cases} \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma \tau^{-1}) & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\ 0 & \text{if } O(\alpha) \neq O(\beta). \end{cases} \tag{2}$$

For  $\alpha \in \Delta$ ,  $V_\alpha^* = \langle e_{\sigma \cdot \alpha}^* \mid \sigma \in G \rangle$  is called the *orbital subspace* of  $V_\chi^n(G)$ , and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^*. \tag{3}$$

Note that it is possible for some  $\alpha \in \Delta$  to have  $V_\alpha^* = 0$ . But Freese [3] proved

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \tag{4}$$

therefore, if we set

$$\bar{\Delta} = \left\{ \alpha \in \Delta \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\},$$

then by (3) we obtain

$$V_\chi^n(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^*. \tag{5}$$

Of course we define the right-hand side of (5) to be 0, if  $\bar{\Delta} = \emptyset$ .

Let  $W$  be a subspace of  $V_\chi^n(G)$ . An orthogonal basis of  $W$  of the form

$$\{e_\alpha^* \mid \alpha \in S\},$$

where  $S$  is a subset of  $I_m^n$ , is called an *O-basis* of  $W$ .

Symmetry classes of tensors associated with subgroups of symmetric groups and their irreducible characters have been studied for a long time. Several papers are devoted to the investigation of the non-vanishing and existence of an *O-basis* for  $V_\chi^n(G)$ , see for example [1,2,5,6,8,9]. In recent years the non-vanishing problem of decomposable symmetrized tensors and so the non-vanishing problem of  $V_\chi^n(G)$ , in the case  $G = S_n$ , have been studied by several authors, see for example [4,7], and the non-vanishing problem in this case has been completely solved (see [4]). But even for  $G = S_n$ , no reasonable result for the structure of *O-basis* of an  $V_\chi^n(G)$  is available. In this note we consider  $G = S_n$  and investigate the existence of an *O-basis* of  $V_\chi^n(S_n)$ , for a special irreducible character of  $S_n$ .

## 2. Main result

Let  $n$  be a positive integer. A *partition*  $\lambda = (\lambda_1, \dots, \lambda_l)$  of  $n$  is a weakly decreasing sequence  $\lambda_1 \geq \dots \geq \lambda_l > 0$  of integers with  $\sum_{i=1}^l \lambda_i = n$ , for short we write  $\lambda \vdash n$ . The number  $l$  is called the *length* of  $\lambda$ , denoted by  $l(\lambda)$ . We often gather together equal parts of a partition and write, for example,  $(5^2, 3^3)$  for  $(5, 5, 3, 3, 3)$ . If  $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ , then  $\lambda' = (\lambda'_1, \dots, \lambda'_s)$ , defined by

$$\lambda'_i = |\{1 \leq j \leq l \mid \lambda_j \geq i\}|,$$

is a partition of  $n$  called the *partition conjugate* to  $\lambda$ .

Frobenius obtained in 1900 an explicit classification of the irreducible complex characters of  $S_n$ ; they are naturally labelled by partitions of  $n$ . We denote the irreducible complex character labelled by the partition  $\lambda$  by  $[\lambda]$ , so the set of all irreducible complex characters of  $S_n$  is  $\text{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$ .

Let now  $V$  be an  $m$ -unitary space. The symmetry class of tensors associated with  $S_n$  and  $[\lambda]$ , where  $\lambda \vdash n$ , is called *full symmetry class of tensors* associated with  $\lambda$  and for short denoted by  $V_\lambda^n$ . Holmes [5] proved that if  $m, n \geq 3$ , then  $V_{(n-1,1)}^n$  is non-zero and does not have an *O-basis*. In this note we consider the special case  $\lambda = (2, 1^{n-2})$ , that is the partition conjugate to  $(n-1, 1)$ ; we prove the analogue of Holmes' result for the space  $V_\lambda^n$ .

The following result was already proved in [7].

**Proposition 2.1.** *Let  $V$  be an  $m$ -unitary space. Let  $\lambda$  be a partition of  $n$ . Then the full symmetry class of tensors associated with  $\lambda$ , i.e.,  $V_\lambda^n$ , is non-zero if and only if  $m \geq l(\lambda)$ . In particular, if  $m \geq n-1$ , then  $V_\lambda^n \neq 0$  for all  $\lambda \neq (1^n)$ .*

We are now ready to state our main result. Note that by Proposition 2.1,  $V_\lambda^n$  is non-zero for  $\lambda = (2, 1^{n-2})$  if and only if  $m \geq n-1$ .

**Main Theorem.** Let  $n \geq 3$  and consider  $\lambda = (2, 1^{n-2})$ . Let  $V$  be an  $m$ -unitary space,  $m \geq n - 1$ . Then the full symmetry class of tensors associated with  $\lambda$ , i.e.,  $V_\lambda^n$ , does not have an O-basis.

### 3. Proof of the Main Theorem

For the proof of the Main Theorem we need a combinatorial result on permutations. We denote by  $\pi$  the natural permutation character of  $S_n$ , i.e., for  $\sigma \in S_n$ ,  $\pi(\sigma)$  is the number of fixed points of  $\sigma$ .

**Lemma 3.1.** Set  $F = \{\sigma \in S_n \mid \pi(\sigma) = \pi(\sigma(1\ 2))\}$ . Then we have  $F = \{\sigma \in S_n \mid \{\sigma(1), \sigma(2)\} \cap \{1, 2\} = \emptyset\}$ .

**Proof.** Let  $\sigma \in S_n$ . There are four possibilities. If  $\{\sigma(1), \sigma(2)\} \cap \{1, 2\} = \emptyset$ , then  $\sigma$  and  $\sigma(1\ 2)$  have exactly the same fixed points. If  $\{\sigma(1), \sigma(2)\} = \{1, 2\}$ , then the number of fixed points of  $\sigma$  and  $\sigma(1\ 2)$  differ by 2. Now assume that  $\{\sigma(1), \sigma(2)\} = \{1, a\}$ , with  $a \neq 2$ . Then in one-line notation the permutations  $\sigma$  and  $\sigma(1\ 2)$  are (not necessarily in that order)  $1\ a\ \sigma(3) \cdots \sigma(n)$  and  $a\ 1\ \sigma(3) \cdots \sigma(n)$ , hence their fixed point numbers differ by 1. The case where  $\{\sigma(1), \sigma(2)\} = \{2, a\}$ , with  $a \neq 1$  is similar. This proves the claim.  $\square$

**Lemma 3.2.** Let  $F = \{\sigma \in S_n \mid \pi(\sigma) = \pi(\sigma(1\ 2))\}$ , and let  $\sigma_1, \dots, \sigma_k$  be distinct permutations of  $S_n$  such that  $\sigma_i \sigma_j^{-1} \in F$  for all  $i \neq j$ . Then  $k \leq [n/2]$ .

**Proof.** Set  $t_r = \sigma_1 \sigma_{r+1}^{-1}$  for  $r = 0, \dots, k-1$ , so  $t_0 = \text{id}$  and  $t_1, \dots, t_{k-1} \in F$ . Then  $t_r^{-1} t_s = \sigma_{r+1} \sigma_1^{-1} \sigma_1 \sigma_{s+1}^{-1} = \sigma_{r+1} \sigma_{s+1}^{-1} \in F$  for all  $r \neq s, r, s \in \{1, \dots, k-1\}$ . Hence by Lemma 3.1,  $\{t_r^{-1} t_s(1), t_r^{-1} t_s(2)\} \cap \{1, 2\} = \emptyset$ , or equivalently,  $\{t_s(1), t_s(2)\} \cap \{t_r(1), t_r(2)\} = \emptyset$  for all  $r \neq s, r, s \in \{1, \dots, k-1\}$ . Thus

$$2k = |\{t_r(j) \mid r = 0, \dots, k-1, j \in \{1, 2\}\}| \leq n$$

and so  $k \leq [n/2]$ .  $\square$

**Remark 3.3.** The bound  $[n/2]$  in Lemma 3.2 is sharp, and the proof shows how to construct such sets of permutations. In particular, we obtain such a set by setting  $\sigma_1 = \text{id}$  and (in cycle notation)  $\sigma_j = (1\ 2j-1)(2\ 2j)$  for  $j = 2, \dots, [n/2]$ .

We are now ready to prove the Main Theorem. Let  $\{e_1, \dots, e_m\}$  be an orthonormal basis of  $V$ . Assume that  $V_\lambda^n$ ,  $\lambda = (2, 1^{n-2})$ , has an O-basis. Put  $\gamma = (1, 1, 2, 3, \dots, n-1)$ , then  $n \geq 3$  and  $m \geq n-1$  implies that  $\gamma \in \Gamma_m^n$ . Consider the action of  $S_n$  on  $\Gamma_m^n$  and choose  $\Delta$  such that  $\gamma \in \Delta$ . It is easy to see that the stabilizer subgroup of  $\gamma$  is equal to  $(S_n)_\gamma = \{(1), (12)\}$ . Therefore

$$\sum_{\sigma \in (S_n)_\gamma} [\lambda](\sigma) = (n-1) + (-1)(n-3) = 2,$$

so it is non-zero and we obtain  $\gamma \in \bar{\Delta}$ . (In fact, this sum is non-zero for any partition  $\lambda \neq (1^n)$ , so  $\gamma \in \bar{\Delta}$  for any  $\lambda \neq (1^n)$ .) Now, by (5), we can decompose  $V_\lambda^n$  into the orthogonal direct sum of orbital subspaces indexed by  $\bar{\Delta}$ . Since  $V_\lambda^n$  has an O-basis, and  $\gamma \in \bar{\Delta}$ , the orthogonality of this decomposition implies that  $V_\gamma^*$  has an O-basis. But, by (4), we have

$$\dim V_\gamma^* = \frac{n-1}{2} \cdot 2 = n-1,$$

so we can assume  $\{e_{g_1 \cdot \gamma}^*, \dots, e_{g_{n-1} \cdot \gamma}^*\}$  is an O-basis for  $V_\gamma^*$ . Therefore for each  $i \neq j$ ,  $1 \leq i, j \leq n-1$ , we have  $\langle e_{g_i \cdot \gamma}^* | e_{g_j \cdot \gamma}^* \rangle = 0$ . On the other hand if  $\alpha = g_i \cdot \gamma$  and  $\beta = g_j \cdot \gamma$ , then  $g_i g_j^{-1} \cdot \beta = \alpha$ , so if we set  $\tau = g_i g_j^{-1}$  and use (2), then we obtain

$$\begin{aligned} \langle e_{g_i \cdot \gamma}^* | e_{g_j \cdot \gamma}^* \rangle &= \frac{[\lambda](1)}{|S_n|} \sum_{\sigma \in (S_n)_\gamma} [\lambda](g_i^{-1} g_j \sigma) \\ &= \frac{n-1}{n!} \left( \varepsilon(g_i g_j^{-1}) (\pi(g_i g_j^{-1}) - 1) \right. \\ &\quad \left. + \varepsilon(g_i g_j^{-1}(12)) (\pi(g_i g_j^{-1}(12)) - 1) \right) \\ &= \frac{n-1}{n!} \varepsilon(g_i g_j^{-1}) (\pi(g_i g_j^{-1}) - \pi(g_i g_j^{-1}(12))), \end{aligned}$$

where  $\varepsilon$  is the sign character. Now the condition  $\langle e_{g_i \cdot \gamma}^* | e_{g_j \cdot \gamma}^* \rangle = 0$ , for each  $i \neq j$ ,  $1 \leq i, j \leq n-1$ , implies that for such  $i, j$  we have  $\pi(g_i g_j^{-1}) = \pi(g_i g_j^{-1}(12))$ . Hence we deduce from Lemma 3.2 that  $n-1 \leq [n/2]$ . Since  $n \geq 3$ , the last inequality is a contradiction and thus  $V_\lambda^n$ ,  $\lambda = (2, 1^{n-2})$ , does not have an O-basis.  $\square$

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**References**

[1] M.R. Darafsheh, M.R. Pournaki, Computation of the dimensions of symmetry classes of tensors associated with the finite two dimensional projective special linear group, Appl. Algebra Engrg. Comm. Comput. 10 (3) (2000) 237–250.  
 [2] M.R. Darafsheh, M.R. Pournaki, On the orthogonal basis of the symmetry classes of tensors associated with the dicyclic group, Linear and Multilinear Algebra 47 (2) (2000) 137–149.

- [3] R. Freese, Inequalities for generalized matrix functions based on arbitrary characters, *Linear Algebra Appl.* 7 (1973) 337–345.
- [4] C. Gamas, Conditions for a symmetrized decomposable tensor to be zero, *Linear Algebra Appl.* 108 (1988) 83–119.
- [5] R.R. Holmes, Orthogonal bases of symmetrized tensor space, *Linear and Multilinear Algebra* 39 (1995) 241–243.
- [6] R.R. Holmes, T.Y. Tam, Symmetry classes of tensors associated with certain groups, *Linear and Multilinear Algebra* 32 (1992) 21–31.
- [7] R. Merris, Non-zero decomposable symmetrized tensors, *Linear Algebra Appl.* 17 (1977) 287–292.
- [8] M.R. Pournaki, On the orthogonal basis of the symmetry classes of tensors associated with certain characters, *Linear Algebra Appl.* 336 (1–3) (2001) 255–260.
- [9] B.Y. Wang, M.P. Gong, The subspace and orthonormal bases of symmetry classes of tensors, *Linear and Multilinear Algebra* 30 (1991) 195–204.